2002

Renewal theory for uniform random variables

Steven Robert Spencer

RENEWAL THEORY FOR UNIFORM RANDOM VARIABLES

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

The thesis answers the question, "How many times must you change a light-bulb in a month if the life-time of any one light-bulb is anywhere from zero to one month in length?" This involves uniform random variables on the interval [0,1] which must be summed to give expected values for the problem. The results of convolution calculations for both the uniform and exponential distributions of random variables give expected values that are in accordance with the Elementary Renewal Theorem and renewal function. There are indications that work on this problem can be traced back to the father of probability theory, Simon Laplace, and Laplace Transforms form a backdrop to the entire investigation.

ACKNOWLEDGMENTS

I would like to express my appreciation to Dr. Charles Stanton, math department, California State University, San Bernardino, for his guidance and encouragement in fulfilling the requirements of this thesis.

To my Aunt Lucy, who gave me my first arithmetic book.

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CHAPTER ONE

INTRODUCTION

The renewal process is a counting process where repeated events occur at random intervals. Each event is independent of other events in the process. Renewal theory, a category of the larger field of probability theory, describes this counting process. It is concerned with finding the number of times a series of random variables can be summed before they reach or exceed a certain number. The function N(t) represents the number of times required in a particular instance and E[N(t)] is the expected value or mean of such a number. This paper will focus on finding formulas for E[N(t)] using one of the classical problems in the discipline first, and then extending the scope of the problem to include overall times greater than the time t in the original problem. The expected values in these cases will be found using the uniform and exponential distributions of random variables.

The best example of the renewal process is found in one of the classical problems of the subject. The so-called "light-bulb problem" asks for the expected value of the number of times a light bulb needs to be replaced in a month if the lifetime of any one bulb is any time from zero to one month in length. This approach to the renewal process uses uniform random variables since the variables are chosen at random on the interval [0,1] and all numbers on the interval have an equal chance of being chosen at any one particular time. Since uniform random variables form the basis for understanding the underlying question involved in this paper, the uniform distribution will be the most closely considered. It also happens that the exponential distribution can be considered because it involves a memoryless property that ties in well with the need for independence of each choice of the random variable.

This light bulb question involves a time limit of one month. If additional times are considered, then it should be possible to find E[N(t)] for values of t (time) for any number of months involved. This will comprise the main focus of this thesis and will involve attempting to find equations representing E[N(t)] for various values of t. Both the uniform and exponential cases will be given and then the final results compared to the values obtained from the Elementary Renewal Theorem and the renewal function. This will provide a general method for evaluating the effectiveness of the chosen approach and will help to establish a criteria for comparison.

General Equation for E[N(t)]

In order to understand the general approach we will take in finding E[N(t)], it will be necessary to understand some basic concepts of the renewal process. Generally speaking, the counting process involved measures the number of time intervals occurring before a certain limiting time t. If $X_1, X_2, \ldots, X_n$ represent various time intervals occurring before the time t, then the waiting time W_2 represents the sum of the first two intervals, $X_1 + X_2$. Also W_n would represent the total waiting time for the sum of all X_K, where K=(1,2,…,n). The probability that a certain X_K would be less than or equal to a specific value, say x, would be written Pr{$X_K \le x$}=F(x), where F(x) is the interoccurrence distribution for the length of an interval. Each X_K value is associated with an event that occurs at the end of each interval. Therefore a change of the kth light bulb in the above problem corresponds to one such event [17].

When dealing with the probability that the waiting time is less than or equal to some value t, it is written Pr{ $W_n \le t\} = F_n(t)$, where $F_n(t)$ is the distribution function for a sum of n numbers. For instance, $W_2 = X_1 + X_2$. To find the probability that $X_1 + X_2 \le a$, where a represents a limiting value, it is necessary to find the cumulative distribution function of $X_1 + X_2$ which can be done in principle by convolution formulas. These formulas will be derived later, but they will give us a method of finding $F_n(t)$ [17].

The renewal counting process for { N(t)} is related to the waiting time process $\{W_n\}$ by understanding that N(t) $\ge k$ if and only if $W_K \le t$. This is the straight forward observation that there are at least k renewals if and only if the total waiting time for k renewals is less than t. This means that Pr$\{N(t) \ge k\} = \Pr\{W_K \le t\} = F_K(t)$. And then, Pr{N(t)=k}$= \Pr\{N(t) \ge k\} - \Pr\{N(t) \ge k+1\} = F_K(t) - F_{K+1}(t)$ [17]. This gives us the link we need to find E[N(t)] since the expected value of a probability function p(y) is defined to be E(Y)$= \sum_y y\, p(y)$ [18]. Therefore

$$E[N(t)] = \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)] \qquad (1).$$

This formula establishes the basic method for finding the expected values we desire and will be used in the calculations throughout this paper.

The problems inherent in this approach are finding formulas for F_n, the distribution function for the sum of n numbers. As was mentioned earlier, the sums of independent random variables can be found in principle by using convolution formulas. Such formulas prove unwieldy, however, for more than three numbers being summed. Another method for finding joint probabilities is the geometric approach. Here too, there are difficulties beyond summing three numbers because for each new number to be added to the sum, the cube takes on another dimension. It becomes difficult to evaluate probabilities beyond the three spatial dimensions.

A better understanding for achieving F_n can be found with delta functions. A suggestion was made of using delta functions as a starting point for computing the derivative of the density function f'(a). The correlation between the two stems from the fact that the delta function integrates at a single point with a value of one and the uniform distribution is f(x)=1 for $0 \le x \le 1$ and 0 otherwise. Therefore the derivative of the density function f can be thought of as equaling $\delta_0 - \delta_1$ where δ_0 and δ_1 are the delta functions at two points, zero and one. A method was found for convolving values for $(\delta_0 - \delta_1)$ and therefore for finding the nth derivative of F_n which is the desired distribution function. Integrating the result n+1 times gives F_n. The method as outlined here will be further described in a following chapter. It turns out that it is indeed possible to derive acceptable results using this method and the results match the algebraic and geometric solutions for up to n=3. This will be the approach to work within and around the confines of the situation.

The remaining sections of the thesis will consider the exponential distribution and its use in the Poisson process. The Poisson process is a counting process and an example of a renewal process. The problems inherent in the uniform distribution do not occur in the exponential case because a series of convolutions for exponential random variables leads to the gamma distribution which is an established distribution and lends itself directly to a value of F_n. Since the exponential distribution can be used for any parameter, it will be possible to compare it with the uniform distribution using a particular value for its calculation.

Finally, these results will be compared to the results expected from the Elementary Renewal Theorem and renewal function to see if an overall pattern can be discerned. This is really the heart of the thesis, to see long term effects of the renewal process as they relate to the original probabilities that were used for the one month time limit. This really provides for a treatment of renewal theory from a short term to long term viewpoint and points out its significance as a trustworthy understanding of this important process.

CHAPTER TWO

LITERATURE REVIEW

The information represented in this thesis is derived almost exclusively from basic concepts and definitions of probability theory and renewal theory. Uniform random variables are described in the introduction of Taylor, An Introduction to Stochastic Modeling, [17] and the exponential distribution and the Poisson process are explained in Chapter 5 of Ross, Introduction to Probability Models [11]. Convolutions of uniform random variables are derived on page 51 of Ross. Application of these variables to renewal theory is found in Chapter 7 of both books. Basically, these two books contain the foundation for most of the material found in this thesis as well as descriptions of the renewal function and Elementary Renewal Theorem. Wackerly, Mathematical Statistics with Applications,[18] gives a treatment of the geometric interpretation of summing uniform random variables which is not found in the other two books. Even with this basic information provided, there seems to be no specific description of finding E[N(t)] for general values of t within these texts. The method employed using delta functions was outside any previously considered method and was developed by Dr. Charles Stanton of California State University, San Bernardino. Delta functions are described in Fourier Transforms by Ian N. Sneddon [15] and are basically the form used in this paper. The application of the delta functions to the present situation and the convolution formulas involving them, were developed separately without literature reference. The fact that $\delta_a * \delta_b = \delta_{a+b}$ for the convolution of a delta function gives rise to the use of binomial coefficients in the distribution functions which are relatively easy to compute. Once a general formula for E[N(t)] has been found using this method, specific values for t can then be described.

The question of finding the expected value of the number of times an independent variable can be summed before it reaches or exceeds a certain time t, has a varied and perhaps long history. Laplace transforms were used in the earliest records of the problem as described in Harry Furstenberg's article in SIAM Review, 1963 [5]. A reference to this particular approach to the problem was unavailable but perhaps points back to Pierre Simon

Marquis de Laplace (1749-1827), the father of modern probability theory. In this article, Furstenberg does derive a specific formula for E[N(t)] and also a result for the Elementary Renewal Theorem as applied to this case. These are the only results for the question considered that were found in the literature.

Two other articles in SIAM Review are written by D.J. Newman and M.S. Klamkin, SIAM,1959,[9] and I.J. Schoenberg, SIAM, 1960 [14]. Both of these articles use W. Weissblum's solution for the expected value using the limiting time as $0 \le t \le 1$. This seems to be the starting point for all investigations of the problem with future discussions centering on values of E[N(t)] for t>1. Since the equation for the distribution function, $F_n = \frac{t^n}{n!} \quad 0 \le t \le 1$, is most easily derived from Laplace transforms, it seems appropriate to assume that Laplace was somehow involved with the result.

The standard text that describes the problem is William Feller, An Introduction to Probability Theory and Its Applications [4]. On pages 26 and 27, Feller describes a distribution function for the sum of independent random variables that is very similar to the one derived in this work. In volume two of Feller's work, a reference was found for the SIAM articles mentioned above which led to the basic information of the history of the problem.

There are other areas that are difficult to discover in the literature as well. The above history made exclusive use of the uniform distribution and did not use the exponential or gamma distributions. The mathematical link between the gamma distribution and the Poisson process was described in Ross, Stochastic Processes, p. 65,[12] and was vital in connecting the exponential distribution with a method for finding F_n using that particular distribution. Once the Poisson process has been established, it is easy to show that E[N(t)] is a particular value. Laplace transforms can also be used in this context, showing a similarity between the uniform and exponential distributions.

Furstenberg's results are equivalent to the results that were obtained by the methods that were used in this thesis and verify our results. The fact that the same results were obtained by entirely different methods serves to confirm both methods and leads to a better understanding of the problem.

CHAPTER THREE

APPLICATION TO UNIFORM RANDOM VARIABLES

Using Equation (1)

This thesis' approach to find E[N(t)] will be to divide the problem into three separate problems. First, the expected value will be found for $0 < t \leq 1$. The other two parts will include finding the solution for t greater than one and also finding a value for E[N(t)] as t increases without bound. In the case of $0 < t \leq 1$, the formula for $F_n(t)$ is given and it is manipulated by equation (1) to provide a solution for this part of the problem. In order to find E[N(t)] for t>1, values of F_n have to be found for each n numbers to be summed. For example if n=2, and we want F_2, that is, two numbers to be summed, we will need to find the joint distribution function for the sum of two independent random variables.

In the case where $0 < t \leq 1$, the distribution function, F_n, is derived from the geometric interpretation of the sum of independent uniform random variables. See equation (4) below. This equation becomes

$$F_n(t)=\frac{t^n}{n!} \qquad \text{since } 0<t\le 1.$$

In order to find E[N(t)], it is necessary to alter equation (1) on page 5 to the following expansion. Note that:

$$\sum_{n=1}^{N} n(F_n - F_{n+1}) = 1(F_1-F_2)+2(F_2-F_3)+3(F_3-F_4)+\ldots+n(F_n-F_{N+1}).$$

So, $\sum_{n=1}^{N} n(F_n - F_{n+1}) = F_1+F_2+\ldots+F_n - NF_{N+1}$.

Therefore, $E[N(t)] = \lim_{N\to\infty}\sum_{n=1}^{N} n(F_n - F_{n+1}) = \lim_{N\to\infty}\sum_{n=1}^{N} F_n - NF_{N+1}$.

We can show $E[N(t)] = \sum_{n=1}^{\infty} F_n$ by proving that $NF_{N+1}\to 0$ as $N\to\infty$.

Intuitively, it can be seen that for any fixed t, the probability that the sum will be less than t approaches zero as the number of variables that are summed increases without bound. A more rigorous proof proves the claim that $F_n(t)\le\frac{t^n}{n!}$. If $\frac{t^n}{n!}$ can be shown to be an upper bound for $F_n(t)$, then (n-1) $F_n(t)$ would go to zero since the factorial term increases faster than the exponential term. This is illustrated geometrically in figure 4 of the Appendix. In the 2-dimensional case, the triangle cut off by the line x+y=a has area G(a) = $\frac{a^2}{2}$. Therefore the area of the shaded region has area no larger than $\frac{a^2}{2}$. Similarly in three dimensions, the simplex under the plane x+y+z=a has volume $\frac{1}{3}\frac{1}{2}a^2a = \frac{a^3}{3!}$. This is therefore an upper bound for the volume illustrated in figure 5. Analogously for a=t, $F_n(t)$ is less than

$$\int_0^t \int_0^{t-x_1} \int_0^{t-x_1-x_2} \cdots \int_0^{t-x_1-x_2\ldots-x_{n-1}} 1 \quad dx_n \ldots dx_3 dx_2 dx_1 = \frac{t^n}{n!}.$$

Therefore $\frac{t^n}{n!}$ represents an upper bound and the following equation holds:

$$E[N(t)] = \sum_{n=1}^{\infty} F_n \qquad (2).$$

This equation will become the one that will be used to find the expected value later on. Using equation (2) and later results, this becomes:

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} - 1 = e^t - 1.$$

This makes use of the Maclaurin series to evaluate the summation sign and represents the solution of the expected value for $0<t\le 1$. In other words, the number of changes of light bulbs in one month is e-1, or about 1.71 changes.

Explicit Convolutions

The above result was obtained because the distribution function, F_n, was known for a time factor of one month or less $(0<t\le 1)$. F_n's are not readily known for times greater than one and these must be calculated. Before a general expression for F_n can be obtained for a specific value of t, the distribution functions such as $F_2, F_3, \ldots$ must be calculated where the subscript notes the number of variables to be summed. This result is commonly found by using convolution equations.

The convolution formula may be derived as follows. Consider $F_{X+Y}(a)$ as the cumulative distribution function of X+Y and so $F_{X+Y}(a)=P\{X+Y\le a\}$. Since we are assuming the distribution of X and Y are independent, and given that f(x) is the probability density of X and g(y) is the probability density of Y, then

$$F_{X+Y}(a)=\iint_{x+y\le a} f(x)g(y)dxdy$$

and so $$\int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f(x)g(y)dxdy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a-y} f(x)dx\right)g(y)dy$$

which is $$\int_{-\infty}^{\infty} F_X(a-y)g(y)dy.$$

This is called the convolution of the distributions F_X and F_Y. The formula that we are using is the probability density function $f_{X+Y}(a)$ and is found by differentiating the distribution function [11]. So

$$f_{X+Y}(a) = \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)g(y)dy = \int_{-\infty}^{\infty} f(a-y)g(y)dy \quad (3).$$

This convolution formula for two functions f and g is written f*g. This formula can be repeated for successive convolutions simply by treating g(y) as the previous convolution result. Using f*f as representing the convolution of a density function with itself, the sum of three random variables, f*(f*f), becomes $\int_{-\infty}^{\infty} f(a-y)(f*f)(y)dy.$

For a single uniform random variable, (uniform on [0,1]) the density function is:

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}.$$

The distribution of the sum of two uniform random variables X and Y is found by using the convolution formula and realizing that g(y) will equal 1 because that is the probability density for one of the two numbers.

Therefore, $f_{X+Y}(a)=\int_0^1 f(a-y)dy.$

Using $f(a-y)=\begin{cases}1 & 0\le a-y\le 1\\ 0 & otherwise\end{cases}$ gives $a-1\le y\le a.$

Since the integrals of the convolution equations depend on the interval concerned, there are two cases, $0\le a\le 1$ and $1<a<2,$ for two uniform random variables. The limits of integration in these cases are found by considering that $a-1\le y\le a$ and that y depends on the interval considered. For $0\le a\le 1,$ $f_{X+Y}(a)=\int_0^a dy=a$ and for $1<a<2$ this yields $f_{X+Y}(a)=\int_{a-1}^1 dy=2-a.$ So,

$$f_{X+Y}(a)=\begin{cases}a & 0\le a\le 1\\ 2-a & 1<a<2\\ 0 & otherwise\end{cases}.$$

This can be represented as illustrated in figure 2 in the Appendix for the density function of X+Y [11].

For the sum of three random variables, the convolution equation becomes:

$$\int_0^2 f(a-y)\,(f*f)(y)dy = \int_0^1 f(a-y)\,ydy + \int_1^2 f(a-y)(2-y)dy.$$

Using the same method for the limits of integration there are three cases. For $0 \le a \le 1$, $\int_0^a ydy = \frac{1}{2}a^2$. For 1<a<2,

$$\int_{a-1}^1 ydy + \int_1^a (2-y)dy = -(a-\frac{3}{2})^2 + \frac{3}{4}.$$

For $2 \le a \le 3$, $\int_{a-1}^2 (2-y)dy = \frac{1}{2}(a-3)^2$. This function can be represented as a quadratic spline and is pictured in figure 3 in the Appendix with the other distributions. This is the density function for the sum of three uniform random variables, $f_{X+Y+Z}(a) = f_3(a)$. Due to the realization that the convolutions become increasingly complex if more numbers are summed, this method is impractical for n>3. Another method will have to be found for knowing F_n.

Geometric Interpretation

To this point the method of determining $F_n(t)$ has been algebraic. There is, however, a geometric method for determining F_n. This method basically provides the same information as the convolution method, that is, F_n up to n=3. The conceptualization becomes difficult after this and F_n's for n>3 are not easy to compute. The geometric method does readily supply general formulas for the first two intervals of a, namely $0 \le a \le 1$ and $1 < a \le 2$, where a is the largest possible value of the sums.

To find F_n for the sum of two independent random variables X and Y, first find their density functions:

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}.$$

The joint density function is

$$f(x,y) = f(x)f(y) = \begin{cases} 1 & 0 \le x \le 1 \ ; 0 \le y \le 1 \\ 0 & elsewhere \end{cases}.$$

The distribution function is $F_{X+Y}(a) = \Pr\{X+Y \le a\}$. We need to find points x and y that satisfy the equation $x+y \le a$. This can be done by considering the line through the square of side one (since x and y are between 0 and 1). The area of the square below the line will be the probability that $x+y \le a$. Therefore $F_{X+Y}(a) = \Pr(X+Y \le a) = \iint\limits_{x+y \le a} f(x,y)dxdy$ [18].

Because $0 \le a \le 2$, the limits of integration will change depending on whether $0 \le a \le 1$ or $1 < a \le 2$. In the case $0 \le a \le 1$, where f(x,y)=1, then $F_{X+Y}(a) = \int_0^a \int_0^{a-y} (1)dxdy = \int_0^a (a-y)dy = \frac{a^2}{2}$. The result may also be obtained by realizing that $F_{X+Y}(a)$ is the volume of the solid with height equal to f(x,y)=1 and the shaded triangle in figure 4. $F_{X+Y}(a) =$ (area of triangle) (height) $= \frac{a^2}{2}(1) = \frac{a^2}{2}$.

For the situation $1 < a \le 2$, the line x+y=a cuts off a triangular corner R from the square (figure 4). The remainder of the square is the probability. The integral becomes $F_{X+Y}(a) = 1 - \iint_R f(x,y)dxdy = 1 - \int_{a-1}^{1} \int_{a-y}^{1} (1)dxdy = \frac{-a^2}{2} + 2a - 1$.

Therefore

$$F_{X+Y}(a) = \begin{cases} 0 & a < 0 \\ \frac{a^2}{2} & 0 \le a \le 1 \\ \frac{-a^2}{2} + 2a - 1 & 1 < a \le 2 \\ 1 & a > 2 \end{cases}$$

Upon differentiating $F_{X+Y}(a)$, the results are $f_{X+Y}(a) = a$ for $0 \le a \le 1$ and $f_{X+Y}(a) = 2 - a$ for $1 < a \le 2$ [18]. These answers are the same as were determined by the algebraic method that was outlined. This basic method is used for all sums, although for each new addition, one more dimension is added to the figure.

The distribution function for the sum of three random variables is analogous to the sum of two such variables. The random variables X,Y,Z have non-zero density over the unit cube. We need to find $F_{X+Y+Z}(a)=\Pr\{X+Y+Z\leq a\}$. We need to find points x,y,z that imply x+y+z<a. For $0\leq a\leq 1$, the graph x+y+z=a is a plane that cuts the three- dimensional axis into a simplex. The values that lead to $x+y+z\leq a$ are found in the volume of the simplex. Volume of simplex = $\frac{1}{3}\frac{a^3}{2}=\frac{a^3}{6}$. The distribution function equals this volume multiplied by the joint density f(x,y,z) over the unit cube which is 1. So $F_{X+Y+Z}(a)=\frac{a^3}{6} \quad 0\leq a\leq 1$ [12].

For $1<a\leq 2$, $F_{X+Y+Z}(a)$ is equal to the volume indicated by a simplex that extends beyond the cube in three smaller simplexes. Three simplexes external to the cube have volume = $3\left(\frac{1}{6}(a-1)^3\right)$ The volume of the remaining cube = $\frac{1}{6}a^3-\frac{1}{2}(a-1)^3$. Therefore $F_{X+Y+Z}(a)=\left(\frac{1}{6}a^3-\frac{1}{2}(a-1)^3\right)(1)=\frac{1}{6}a^3-\frac{1}{2}(a-1)^3$.

For $2 \le a < 3$, $F_{X+Y+Z}(a)$ is the cube remaining after the simplex using the plane x+y+z = 3 is cut off. One dimension of the simplex = 1-(a-2) = 3-a. The volume equals $\frac{1}{6}(3-a)^3$ and the complementary volume is $1-\frac{1}{6}(3-a)^3$. $F_{X+Y+Z}(a) = 1-\frac{1}{6}(a-3)^3(1) = 1-\frac{1}{6}(3-a)^3$. [16] See figure 5 in Appendix for illustrations.

Using $X_1, X_2, \ldots X_n$ as random variables, the interval $0 \le a \le 1$ can be described in a variety of ways using various sums. $\Pr\{X_1 + X_2 \le a\} = \int_0^a \int_0^{a-x_1} 1 dx_2 dx_1 = \frac{a^2}{2}$. Also,

$\Pr\{X_1 + X_2 + X_3 \le a\} = \int_0^a \int_0^{a-x_1} \int_0^{a-x_1-x_2} 1 dx_3 dx_2 dx_1 = \frac{1}{6}a^3$. Four variables follow the pattern.

$\Pr\{X_1 + X_2 + X_3 + X_4 \le a\} = \int_0^a \int_0^{a-x_1} \int_0^{a-x_1-x_2} \int_0^{a-x_1-x_2-x_3} 1 dx_4 dx_3 dx_2 dx_1 = \frac{1}{24}a^4$. Thus, the general formula appears to be:

$$F_n(a) = \frac{a^n}{n!} \Rightarrow f_n(a) = \frac{a^{n-1}}{(n-1)!} \qquad (4).$$

This gives a general expression for the distribution and density functions in the interval $0 \le a \le 1$, but not for succeeding intervals.

The geometric solution for $1 < a \le 2$ is found by seeing that when the simplex concept is extended to more dimensions, the case for n = 3 when $1 < a \le 2$, $\frac{1}{6}a^3 - 3\left(\frac{1}{6}(a-1)^3\right)$, can be considered as $\frac{1}{n!}a^n - n\left(\frac{1}{n!}(a-1)^n\right)$. This result will compare with the answers obtained with the delta functions technique for the distribution functions calculated. The geometric results help to confirm the previous methods and are an important part of finding F_n. Therefore we have found F_n for $n \le 3$ and general F_n for the first two intervals considered, $0 < a \le 1$ and $1 < a \le 2$. We will now attempt to find a general formula for F_n that takes into consideration all the intervals involved. Once a general formula for F_n is found, then it should be straightforward to find E[N(t)].

Delta Functions

An alternative approach to produce F_n's for large n's is to make use of delta functions. Delta functions make use of the fact that it is possible to integrate at a single point (a), on a line. If the delta function, δ_a, is zero everywhere except for a very small portion about a, then $\int_{-\infty}^{\infty} \delta_a = 1$ [15]. Because the continuous function f has a nearly constant value close to f(a) on this interval,

$$\int_{-\infty}^{\infty} f(x)\delta_a(x)dx \cong f(a) \int_{small\ portion} \delta_a(x) = f(a).$$

If δ_a is considered to be an abbreviation for dF_a, where

$$F_a(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases},$$

then $F_a' = \delta_a$. By considering the density for the random variable uniform on the interval [0,1]:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}.$$

Then df can be considered as $\delta_0 - \delta_1$. Delta functions make use of Stieltjes integrals and many of the same theorems are carried over from Riemann integrals. Basically the function was invented by physicists in the late nineteenth century to solve problems dealing with physical events at a point in time.

The delta functions may be applied to the convolution system as follows. In understanding an algebra of convolutions, it is true that f*g = g*f, where * represents the convolution calculation. This can be seen by writing

f*g = $\int_{-\infty}^{\infty} f(a-y)g(y)dy$. Let u=a-y, y=a-u, and dy=-du. Then the

above equation equals $\int_{-\infty}^{\infty} f(u)g(a-u)du$. Also the relationship

(f*g)′ = f′*g is shown by

$$\frac{d}{da} f_{X+Y}(a) = (f*g)' = \frac{d}{da}\int_{-\infty}^{\infty} f(a-y)g(y)dy$$

and $$\int_{-\infty}^{\infty} \frac{d}{da} f(a-y)g(y)dy = \int_{-\infty}^{\infty} f'(a-y)g(y)dy = f'*g.$$

Also (f*g)′=(g*f)′=g′*f. And (f*f)″=f′*f′=f″*f. By changing

the variables as above, $\int_{-\infty}^{\infty} f'(a-y)f(y)dy = \int_{-\infty}^{\infty} f'(u)f(a-u)du$ and

$\frac{d}{da}\int_{-\infty}^{\infty} f'(u)f(a-u)du = \int_{-\infty}^{\infty} f'(u)f'(a-u)du.$ Therefore, the nth

derivative of the n-fold convolution, $(f*f*f*\ldots*f)^{(n)}$ =

f′*f′*…*f′ n times. Since we are using f′=$\delta_0 - \delta_1$ from our

understanding of delta functions, we have a method of

calculating f*…*f by starting with delta functions and

integrating n+1 times. What is missing is a convolution

formula for delta functions so that the integrations can

take place [16].

In order to come to a definition of the convolution of a δ-function, we consider the integral $\int_{-\infty}^{\infty} h(x)\,[f*g(x)]\,dx$ with h(x) a continuous function that is everywhere differentiable. Using the convolution formula, this becomes: $\int_{-\infty}^{\infty} h(x)\left\{\int_{-\infty}^{\infty} f(x-y)g(y)dy\right\}dx.$ Interchanging the order of integration, this becomes: $\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} h(x)f(x-y)dx\right\}g(y)dy.$ Setting u = x-y, du = dx, x = u+y, this becomes:

$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} h(u+y)f(u)du\right\}g(y)dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u+y)f(u)g(y)dudy.$ Therefore,

$\int_{-\infty}^{\infty} h(x)\,[f*g(x)]\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u+y)f(u)g(y)dudy$ suggests the definition for convolution of the δ-function. $\delta_a * \delta_b$ is defined by:

$\int_{-\infty}^{\infty} h(x)\,[\delta_a * \delta_b](x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u+y)\delta_a(u)\delta_b(y)dudy.$ This equation equals $\int_{-\infty}^{\infty} h(a+y)\delta_b(y)dy = h(a+b) = \int_{-\infty}^{\infty} h(x)\delta_{a+b}(x)dx,$ so

$$\delta_a * \delta_b = \delta_{a+b} \qquad (5).$$

Therefore by finding $(\delta_0 - \delta_1)^{*n}$ for n = 2,3, etc., it is possible to find F_n. The process involved will amount to a convolution of the delta function followed by n+1 integrations which will arrive at F_n where n represents the number of random numbers to be summed. Then E[N(t)] can be found using equation (1) or equation (2) [16].

General Formula for E[N(t)]

Using the formula $(\delta_0 - \delta_1)^{*2}$ for n=2, there is

$$(\delta_0 - \delta_1) \ * \ (\delta_0 - \delta_1) =$$

$$\delta_0 \ * \ \delta_0 - \delta_1 \ * \ \delta_0 - \delta_0 \ * \ \delta_1 + \delta_1 \ * \ \delta_1 = \delta_{0+0} - \delta_{1+0} - \delta_{0+1} + \delta_{1+1} = \delta_0 - 2\delta_1 + \delta_2.$$

With n=3, $(\delta_0 - \delta_1)^{*3} = \delta_0 - 3\delta_1 + 3\delta_2 - \delta_3$. As can be seen, the coefficients of the delta functions are the same as binomial coefficients. This is because of the similarity between $(\delta_0 - \delta_1)^{*n}$ and $(a-b)^n$. Since the convolution operation is also distributive, the coefficient of an individual delta function found in this way will be $(-1)^k \binom{n}{k} \delta_k$. The total form of the delta equation for a particular value of n will be

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\delta_k \qquad (6).$$

Therefore $F_n(x)=\int^{(n+1)}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\delta_k$. It is possible to introduce a new function at this point which will specifically relate δ_k to a form that will be able to compute E[N(t)]. This is called the plus-function and is defined by:

$$x^+=\begin{cases} x & x\geq 0 \\ 0 & x<0 \end{cases}.$$

If we follow the results of n+1 integrations of a particular delta function, we get the following results.

For $$\int_{-\infty}^{x}\delta_a=\begin{cases} 0 & x<a \\ 1 & x>a \end{cases},$$

the value of the integral depends on whether x is more or less than a. If there is a second integration,

$$\int_{-\infty}^{x}\int_{-\infty}^{x}\delta_a=\int_{0}^{x}1=\begin{cases} (x-a) & x>a \\ 0 & x<a \end{cases} \qquad \Rightarrow (x-a)^+ .$$

A third integration of a particular delta function leads to:

$$\int_{-\infty}^{x}\int_{-\infty}^{x}\int_{-\infty}^{x}\delta_a=\begin{cases} 0 & x<a \\ \dfrac{(x-a)^2}{2} & x>a \end{cases} \qquad \Rightarrow \frac{\left[(x-a)^+\right]^2}{2} .$$

Following this procedure,

$$\int^{(n+1)} \delta_a = \frac{\left[(x-a)^+\right]^n}{n!} .$$

Putting this in the above expression of $F_n(x)$, it becomes

$F_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \left[(x-k)^+\right]^n}{n!}$. Allowing x to equal t, this gives us $F_n(t)$:

$$F_n(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\left[(t-k)^+\right]^n}{n!} \qquad (7).$$

Since the expected value is found from equation (2), it will be necessary to find F_0 in order to perform the double summation sign that will be involved. For us, $F_0 = 1$, which actually indicates that the first light bulb present at n=0 is being included. The expected value equation now becomes:

$$E[N(t)+1] = \sum_{0}^{\infty} F_n \qquad (8).$$

Using equations (7) and (8), this becomes:

$$E[N(t)+1] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\left[(t-k)^+\right]^n}{n!} \qquad (9).$$

We will use the formula for interchanging the order of summation:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{nk} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} c_{nk} .$$

Equation (9) becomes:

$$E[N(t)+1] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} . \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \frac{\left[(t-k)^+\right]^n}{n!} .$$

Because $(t-k)^+ = 0$ if k>t, we can write $\sum_{k=0}^{\lfloor t \rfloor} \sum_{n=k}^{\infty} c_{nk}$. This will finally lead to our final equation for E[N(t)+1]:

$$E[N(t)+1] = \sum_{k=0}^{\lfloor t \rfloor} (-1)^k e^{t-k} \frac{(t-k)^k}{k!} \qquad (10).$$

So the renewal function is:

$$M(t)=E[N(t)] = \sum_{k=0}^{\lfloor t \rfloor} (-1)^k e^{t-k} \frac{(t-k)^k}{k!} -1.$$

Using equation (10) to derive values for t=2,3 and 4, we have the following:

t=2	$E[N(t)+1] = e^2 - e =$	4.671
t=3	$E[N(t)+1] = e^3 - 2e^2 + \frac{e}{2} =$	6.6665
t=4	$E[N(t)+1] = e^4 - 3e^3 + 2e^2 - \frac{e}{6} =$	8.6667.

By subtracting 1 from each entry, it would equal E[N(t)].

Elementary Renewal Theorem

So far, we have investigated E[N(t)] for specified time intervals. There is a basic tenant of renewal theory that focuses on E[N(t)] for $t \to \infty$. This would give the results for large values of t. It is called the Elementary Renewal Theorem [17] and is denoted as:

$$\lim_{t\to\infty}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{\mu}.$$

The symbol μ is the mean interoccurrence time and is written $E[X_K]$. In a sense, this result can seem obvious. Since E[N(t)] is the mean lifetime of a component, division by t would give the lifetime per unit time, which would be the reciprocal of the mean. Actually, detailed calculations are necessary to establish this.

In the uniform case, $E[X_K]=\frac{1}{2}$ because the mean value over the interval [0,1] for a single variable would be midway between the endpoints of the interval. The Elementary Renewal Theorem would give $\frac{1}{\mu}=\frac{1}{\frac{1}{2}}=2$ or 2t if measured by the total time t in the system. This compares favorably with our values for E[N(t)] which are always approximately one-third of a unit below the value of 2t. Our values reinforce the equation found in Furstenberg's article [5] that indicates

CHAPTER FOUR

APPLICATION TO EXPONENTIAL RANDOM VARIABLES

It is also possible to apply renewal theory to exponential random variables. The exponential case is useful for us because its random variables are also independent and in that sense are analogous to the uniform case. It turns out that the exponential distribution gives rise to the gamma distribution that provides an easy access to F_n just as the delta functions helped us with the uniform case.

In order to find F_n for the exponential case, it is necessary to give a definition for an exponential distribution for a continuous random variable. This definition uses a parameter λ with $\lambda > 0$. The probability density function is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Before we show how this exponential function gives rise to F_n, it is necessary to understand something of the Poisson process which will ultimately be used to find E[N(t)].

First we will define a counting process in general and then show how it corresponds to the Poisson process.

A general definition is taken from Ross, Introduction to Probability Models:[11]

> A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a counting process if N(t) represents the total number of "events" that have occurred up to time t. From its definition we see that for a counting process N(t) must satisfy:
> (i) $N(t) \geq 0$
> (ii) N(t) is integer-valued.
> (iii) If s<t, then $N(s) \leq N(t)$.
> (iv) For s<t, N(t)-N(s) equals the number of events that have occurred in the interval (s,t].

There are two concepts that are inherent in this definition. Independent increments in a counting process means that once a choice is made for a random number it does not affect future choices. Stationary increments indicate that the number of events in an interval depend only on the length of the interval. If two time intervals are equal, they must have the same probabilities for the number of events connected with them.

Definition 3.1 in Ross states:[11]

> The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if
> (i) N(0)=0
> (ii) The process has independent increments.
> (iii) The number of events in any interval of length t is Poisson distributed with mean λt.

That is, for all

$s,t \geq 0, \quad P\{N(t+s)-N(s)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n=0,1...$

Note that it follows from condition (iii) that a Poisson process has stationary increments and also that E[N(t)]=λt which explains why λ is called the rate of the process.

We are now in a position to calculate F_2 by using the convolution formula derived before and a knowledge of the interoccurrence distribution F. Since we know how the exponential function is defined, $f(x)=\lambda e^{-\lambda x}$, then $F(x)=1-e^{-\lambda x}$ for $x \geq 0$, which is the interoccurrence distribution. So using the formula

$$F_n(x)=\int_0^\infty F_{n-1}(x-y)dF(y)=\int_0^x F_{n-1}(x-y)dF(y), \quad [17]$$

and starting with $F_{X_1+X_2}(t)=P\{X_1+X_2 \leq t\}$, where t and s can be used for x and y in the above equation, this equation is

$$\int_0^t P\{X_1 \leq t-s\}\lambda e^{-\lambda s}ds=\int_0^t \left[1-e^{-\lambda(t-s)}\right]\lambda e^{-\lambda s}ds$$

which gives $1-e^{-\lambda t}-\lambda te^{-\lambda t}$.

If this is differentiated it becomes $f_{X_1+X_2}(t)=\lambda^2 te^{-\lambda t}$, $t \geq 0$.

This is the gamma distribution with parameter 2 and λ.

Ross also states, "In general it turns out that if $X_1, X_2, ..., X_n$ are independent and identically distributed exponential random variables having mean $\frac{1}{\lambda}$, then $X_1 + X_2 + \ldots + X_n$ has a gamma distribution with parameters n and λ [11]." This is the connection between F_n, interoccurrence time, and the gamma distribution. In this sense the gamma distribution is used in the same way as the delta functions for the uniform case because it serves as a simple and direct way to find F_n by first finding f_n. The gamma distribution is written $f_n = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$ as a density function with parameter λ.

This can be broken up as follows by using a telescoping series. The gamma distribution equals:

$$\lambda e^{-\lambda t}\left\{\sum_{j=n}^{\infty}\left[-\frac{(\lambda t)^j}{j!}+\frac{(\lambda t)^{j-1}}{(j-1)!}\right]\right\} = -\sum_{j=n}^{\infty}\lambda e^{-\lambda t}\frac{(\lambda t)^j}{j!}+\sum_{j=n}^{\infty}\lambda e^{-\lambda t}\frac{(\lambda t)^{j-1}}{(j-1)!}$$

which gives: $f_n = \frac{d}{dt}\sum_{j=n}^{\infty} e^{-\lambda t}\frac{(\lambda t)^j}{j!}$.

The integral of this term will give $\Pr\{S_n \le t\}$ or $F_n = \sum_{j=n}^{\infty} e^{-\lambda t}\frac{(\lambda t)^j}{j!}$ [12]. Since $\Pr\{N(t) = n\} = F_n - F_{n+1}$, then $\Pr\{N(t) = n\}$ will equal $\frac{(\lambda t)^n}{n!}e^{-\lambda t}$, which is the Poisson distribution with parameter λ. Once it is determined that a Poisson process is involved, the finding of E[N(t)] is very straightforward.

$$E[N(t)] = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t}(\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = \lambda t \qquad (12).$$

Equation (12) is called the renewal function for the exponential distribution and is written:

$$M(t)=E[N(t)]= \lambda t .$$

Since the mean of the exponential distribution E[X], is given by $E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} \lambda x e^{-\lambda x} dx = \frac{1}{\lambda}$, then $\mu = \frac{1}{\lambda}$ and $\frac{E[N(t)]}{t} = \frac{1}{\mu}$. Since the mean, μ, is a function of λ, a constant, μ is also constant. No limit is necessary in this case and so the equation derived from the renewal function is equivalent to the Elementary Renewal Theorem. By using $\mu = \lambda^{-1}$ in equation (12), the result is also the same as that derived from the renewal function and helps to confirm the process of arriving at equation (12).

Since λ is a parameter of the renewal function, it can be adjusted to reflect the condition desired. By setting $\lambda = 2$, the mean, μ, becomes $\frac{1}{2}$ and this corresponds to an interoccurrence time of the exponential distribution that is the same as for the uniform case. It is now possible to see that the expected value for the exponential case equals 2t which is identical to the result of the Elementary Renewal Theorem. Thus the exponential case can be shown to reinforce the results that were already found for the uniform case.

CHAPTER FIVE

SUMMARY

As was mentioned at the end of Chapter Three, the source of equation (11) as found in Furstenberg's article will be discussed in this final chapter. Both in the Schoenberg article and the Furstenberg article Laplace Transforms were mentioned as a possible alternative method for understanding results such as equation (11). Schoenberg goes on to relate that George Polya's doctorial thesis was an intensive investigation of Laplace Transforms related to problems similar to ours [14].

Was Laplace himself somehow connected with the probability theory that has led to Renewal Theory? There are conflicting views on Laplace's part in developing Laplace Transforms, yet it is known that he used this integral transform when developing his theories on probability. It is conceded that Euler first discovered the integral yet Laplace's work on probability made use of it [20]. The first edition of Laplace's great work on probability, <u>Theorie Analytique des Probabilities</u>, was published in 1812 and contains methods of finding probabilities of compound events when the probabilities of these simple components are known [10]. Laplace used characteristic functions to represent the random variables [13]. Workers after Laplace further developed the transforms, most notably Jozeph Petzval [3]. The Laplace Transform in its present form differs slightly from the earlier version yet they are both used to solve differential equations [2]. Laplace was certainly one of the first to make use of the integral and it is fitting that his name should be attached to it.

Perhaps the most basic realization that Laplace Transforms are involved with the type of probability situations that are considered in this paper stem from the convolution equation itself. The convolution equation basically deals with the probability involved when summing independent random variables. If two functions are involved in the convolution equation, the results of the convolution are the products of the Laplace Transforms of those two functions. $L\{f(t)\}$ represents the Laplace Transform of f(t). Using the definition of the Laplace Transform,

$$L\{f(t)\}=\int_0^\infty e^{-st} f(t)dt = F(s),$$

the convolution equation becomes:

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau \quad \text{and} \quad L\{f * g\} = F(s)G(s) \quad [20]$$

since there are two functions involved in the convolution formula.

Such a basic relationship would seem to indicate that the very purpose of the Laplace Transforms was directed at solving the convolution equation. The question seems to be, can the Laplace Transforms be used to find a general expression for F_n? By defining the density function f(t) of a uniform random variable over the interval [0,1] we have:

$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & otherwise \end{cases}.$$

This gives $L\{f(t)\}$ as

$$\int_0^1 e^{-st} 1 dt + \int_1^\infty e^{-st} 0 dt \;=\; -\frac{1}{s} e^{-st} \Big|_0^1 = \frac{1}{s}\left(1 - e^{-s}\right).$$

Using the formula for the convolutions with Laplace Transforms there is:

$$L\{f^{*n}\} = \left[\frac{1}{s}\left(1 - e^{-s}\right)\right]^n.$$

Expanding the right-hand side using the binomial theorem

gives $L\{f^{*n}\} = \frac{1}{s^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{-sk}.$

Using $L\{f^{*n}\} = L\{F_n'\} = sL\{F_n\}$,

then $L\{F_n\} = \dfrac{1}{s^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{-sk}$.

It is understood that by taking the inverse transform of the above expression, the desired distribution function is obtained. In comparing the above formula with equation (7), it becomes apparent that the following condition must hold for equality to exist between this Laplace Transform method and the earlier delta function method:

$$L^{-1}\left\{\frac{e^{-as}}{s^{n+1}}\right\} = \frac{\left[(t-k)^+\right]^n}{n!}.$$

Is there an inverse transform that would allow such a condition to exist?

Given $$L^{-1}\left\{\frac{e^{-as} n!}{s^{n+1}}\right\} = (t-a)^n U(t-a) = \left[(t-a)^+\right]^n \qquad t > 0$$

where $U(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$,

the inverse transform produces the desired result if the n! term is brought outside the inverse transform and then allowed to become the denominator of the right-hand term. Therefore it has been shown that Laplace Transforms produce the same result as was earlier determined by the delta function method.

Whether Furstenberg used this approach to determine his equation using Laplace Transforms is unclear yet the Transforms do provide a straightforward method of finding n convolutions of delta functions which are converted back to an expression for F_n.

The fact that succeeding researchers made use of Laplace Transforms for the probability questions asked in this paper show the usefulness of this approach. Although Renewal Theory is twentieth century development, roots can be traced to Laplace himself who is called the "father of modern probability theory [19]."

Renewal Theory accurately models many biological, economic, and scientific processes. Therefore, natural processes often lend themselves to this theory. Laplace believed in a universe that was governed by complete order and his theories of probability were intended to show that even events that "did not seem to follow the grand laws of nature"[8] were the result of it just the same. In the introduction to his great work on probability, Laplace states, "so that the entire system of human knowledge is connected with the theory set forth in this essay [8]." With such a grand scope to consider, it is evident that

Laplace considered nothing as outside of the theory that he proposed. Few men would attempt such a feat and fewer still would attain to it. The result that so much of modern probability theory owes its beginnings to this man shows that his ideas were far-reaching and that he succeeded in establishing a new science that finds its way into many aspects of our lives. Laplace, perhaps, cannot be considered the author of Renewal Theory, yet his work certainly laid a foundation that resulted in this theory.

APPENDIX

Figure One: Density Functions for

the Sum of Uniform Independent

Random Variables

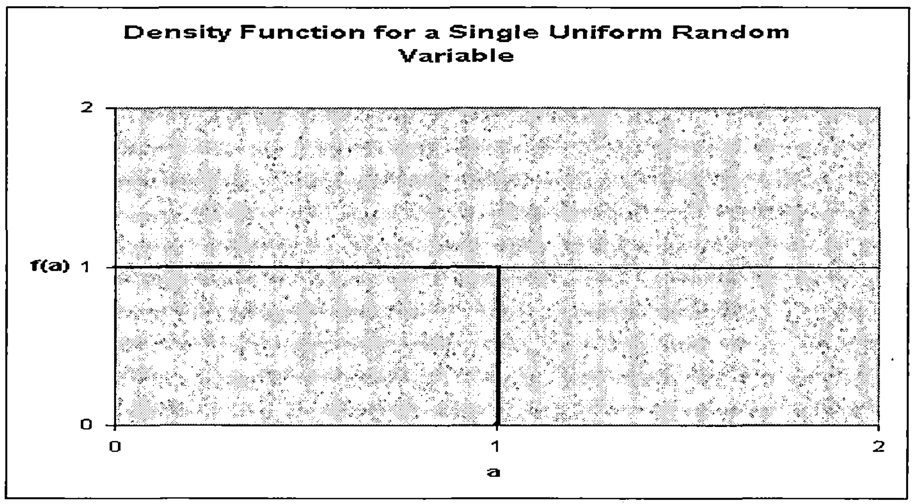


Figure Two: Density Function for the Sum of Two Uniform Random Variables

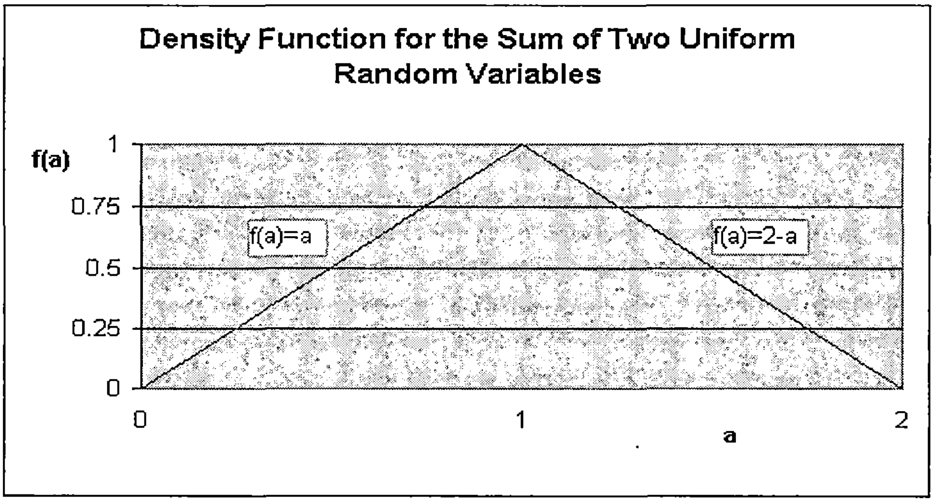


Figure Three: Density Function for the Sum of Three Uniform Random Variables

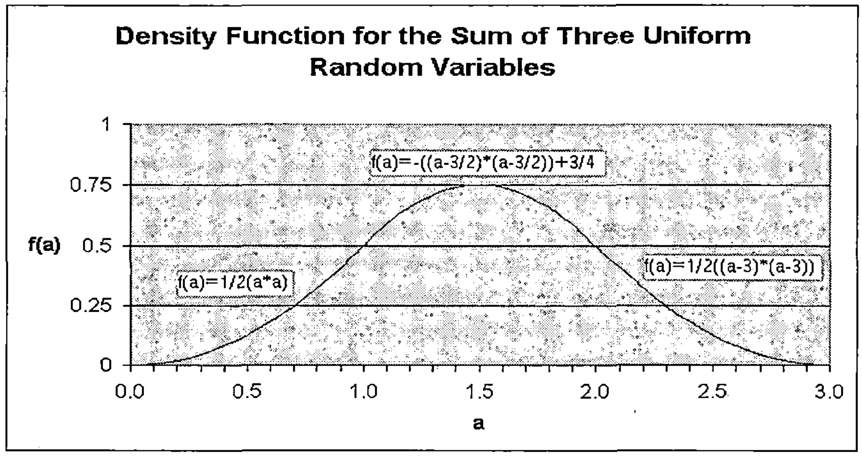
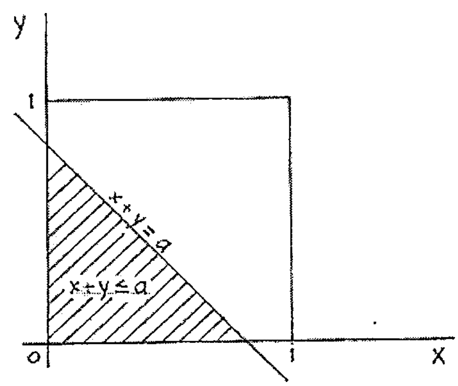
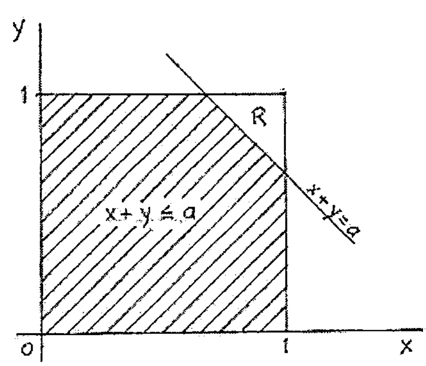


Figure Four: Geometric Interpretation for the Sum of Uniform Independent Random Variables

Geometric Interpretation for the Sum of
Two Uniform Independent Random Variables



Shaded Region for 0≤a≤1



Shaded Region for 1≤a≤2

Figure Five: Geometric Interpretation

for the Sum of Uniform Independent

Random Variables

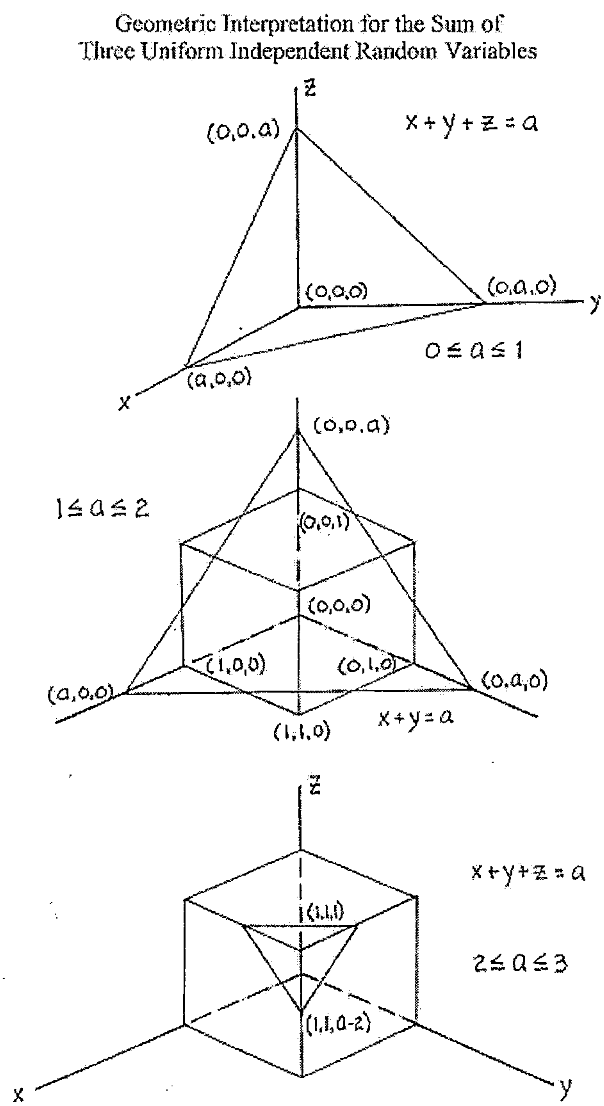
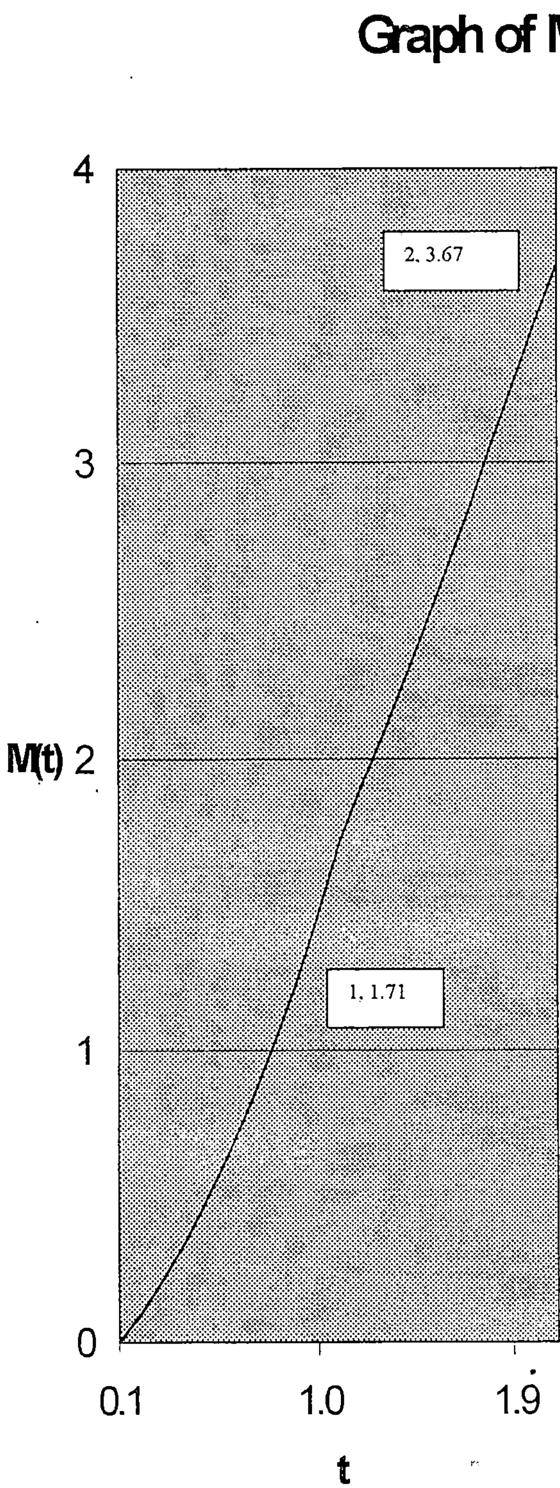


Figure Six: Result For Renewal Function



$$M(t) = \begin{cases} e^t - 1 & 0 < t \le 1 \\ e^t - (t-1)e^{t-1} - 1 & 1 < t \le 2 \end{cases}$$

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